Finite Math - Fall 2018

Lecture Notes - 10/9/2018

Homework

- Section 1.2 - 5, 6, 7, 8, 23, 25, 27
- Graph the following lines:
  (1) $y = 2x$
  (2) $2x + 3y = 0$
  (3) $5x - 6y = 0$
- Section 4.1 - 1, 5, 7, 9, 10, 11, 12, 13, 17, 20, 21, 23, 25, 26, 27, 28, 31, 33

Section 1.2 - Graphs and Lines

Definition 1 (Line)**.** *A* line *is the graph of an equation of the form*

$$Ax + By = C$$

where not both of A and B are equal to zero (i.e., if $A = 0$, then $B \neq 0$ and vice-versa).

Graphing Lines. There are two common ways of graphing lines: by **finding intercepts** and by **using the slope and a point**. We will focus on the method of finding intercepts here in the notes. You can read about using the slope to graph a line in the textbook.

Finding Intercepts.

Definition 2 (Intercept)**.** *A point of the form $(a, 0)$ on a line is called an x-*intercept *and a point of the form $(0, b)$ is called a y-*intercept*.*

Every line will have at least one intercept, but most have two. There are three special cases in which the line has only one intercept: if $A = 0$, $B = 0$, or $C = 0$. We will return to these special cases in a little bit.

Assume the line $Ax + By = C$ has both an x- and $y-$ intercept, we find them as follows:

- To find the x-intercept, we set $y = 0$ in the equation of the line and solve for x. Symbolically, this means that

$$x = \frac{C}{A}.$$

- To find the y-intercept, we set $x = 0$ in the equation of the line and solve for y. Symbolically, this means that
$$y = \frac{C}{B}.$$

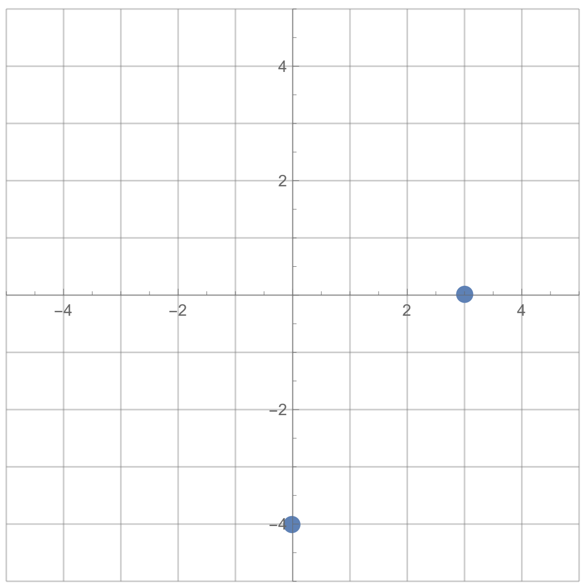
To graph a line using intercepts, we plot the two intercepts in the xy-plane, and draw a line through the points:

Example 1. *Graph the line $4x - 3y = 12$ using intercepts.*

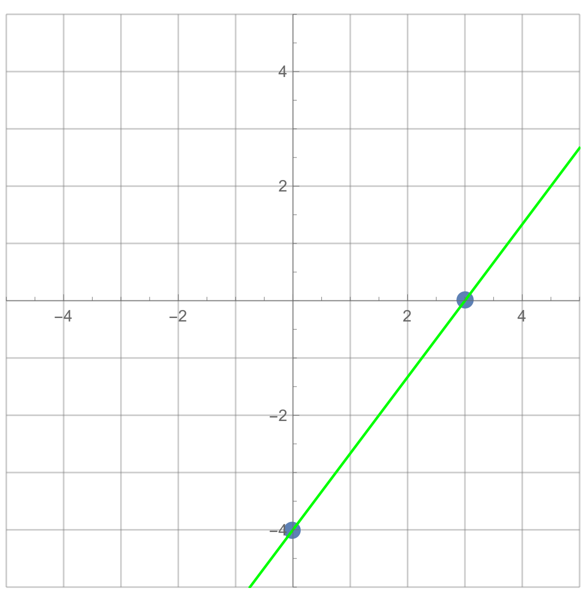
Solution. *First find the intercepts:*

- *set $x = 0$ to get $-3y = 12$ so $y = -4$. Thus the y-intercept is $(0, -4)$.*
- *set $y = 0$ to get $4y = 12$ so $x = 3$. Thus the x-intercept is $(3, 0)$.*

Now plot these points:



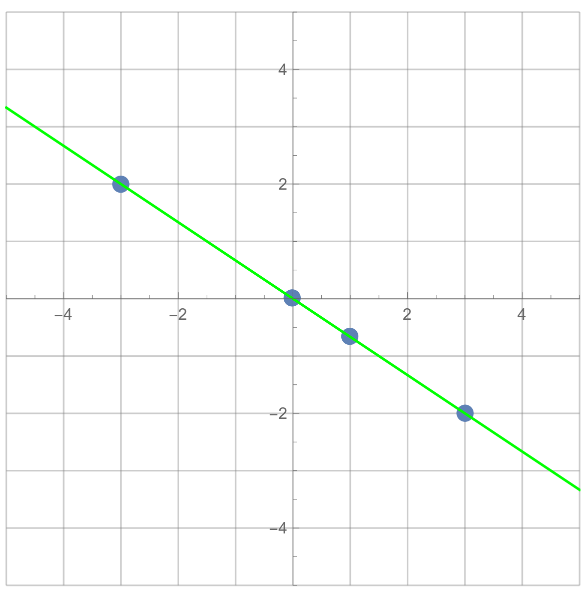
Then we draw a line through them:



Now, let's talk about one of those special cases, when $C = 0$ in $Ax + By = C$. We will assume that $A, B \neq 0$ here. If $C = 0$, you'll find that solving for the x-intercept as above gives $(0, 0)$ and solving for the y-intercept also gives $(0, 0)$. This means that both the x- and y- intercepts are at the origin. So, to graph the line $Ax + By = 0$, we need to come up with another point. You can really just pick any number other than 0 for x or y, then solve for the opposite. One easy thing that always works is to use one of the points $(B, -A)$ or $(-B, A)$, both are points on the line (check this!). That is, you just take the coefficients of x and y, flip their order, and put a minus sign in front of one of them. As a matter of fact, you could just use the two points $(B, -A)$ and $(-B, A)$ to graph the line. There's many other points you could use (which might be simpler than the previous two), but these always work.

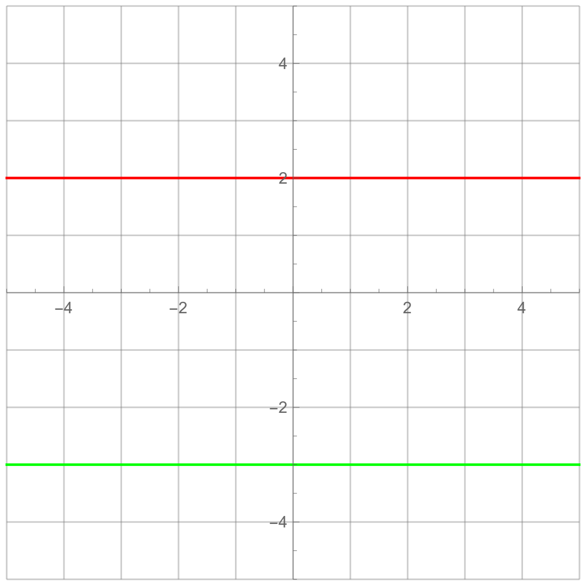
Example 2. *Graph the line* $2x + 3y = 0$.

Solution. *If we try to solve for either intercept on this line, we will get the point* $(0, 0)$. *We need to come up with a second point now to actually graph the line... We can use the trick from above to come up with the point* $(-3, 2)$ *or* $(3, -2)$. *An alternate way we could find a point is, say, to set* $x = 1$. *Then the equation for the line gives us* $2(1) + 3y = 2 + 3y = 0$, *and solving for* y *gives* $y = -\frac{2}{3}$. *So, we also have the point* $\left(1, -\frac{2}{3}\right)$ *on the line. We'll plot all of these points, just so we can see that they are all on the line, but we could get away with just any two of them.*

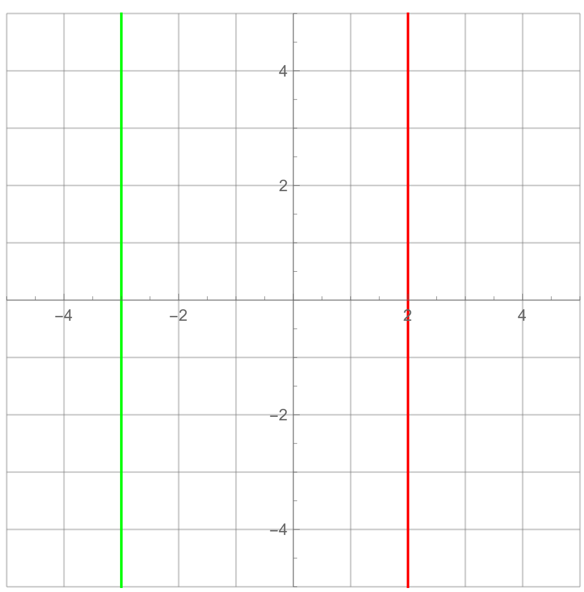


Horizontal and Vertical Lines. The cases when $A = 0$ or $B = 0$ in $Ax + By = C$ correspond to horizontal and vertical lines, respectively.

- If $A = 0$, we end up with the line $y = \frac{C}{B}$, which is a horizontal line where every y-value is $\frac{C}{B}$. A special one of these is when C is also zero so we get the equation $y = 0$. The graph of this line is the x-axis. Here are the graphs of $y = 2$ (red) and $y = -3$ (green).



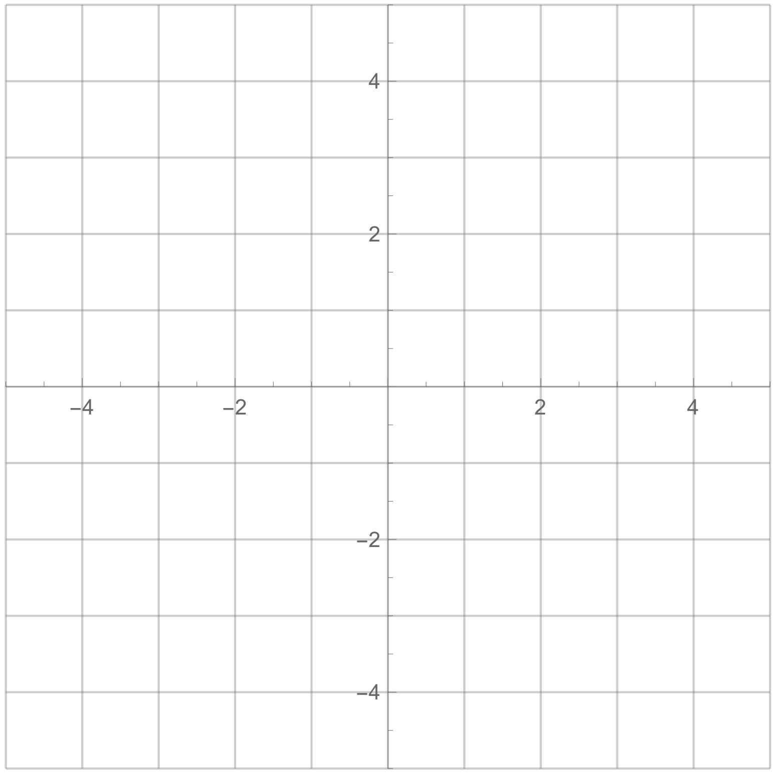
- If $B = 0$, we end up with the line $x = \frac{C}{A}$, which is a vertical line where every x-value is $\frac{C}{A}$. A special one of these is when C is also zero so we get the equation $x = 0$. The graph of this line is the y-axis. Here are the graphs of $x = 2$ (red) and $x = -3$ (green).



Example 3. *Graph the following lines:*

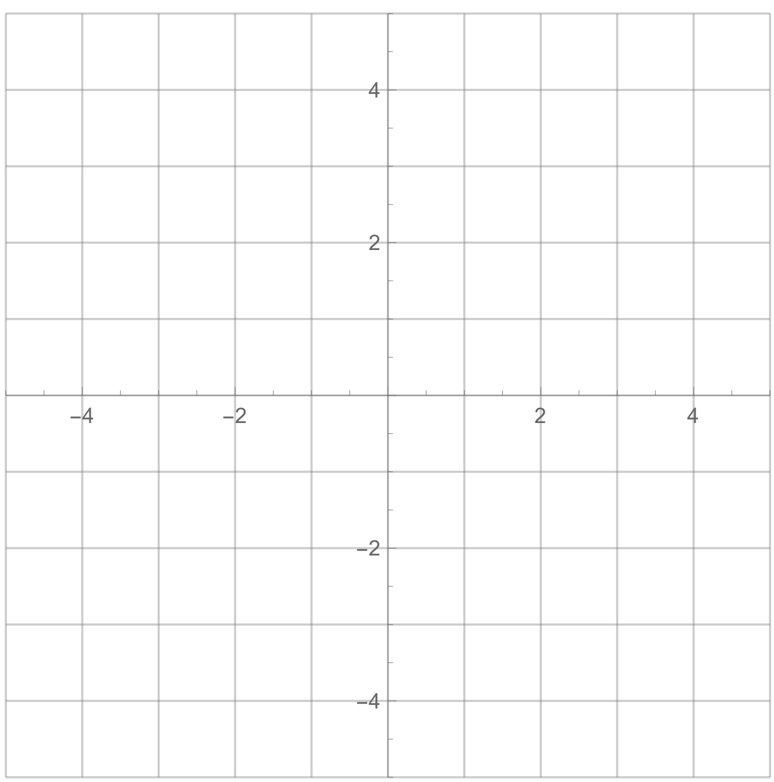
(a)

$$2x - y = 3$$



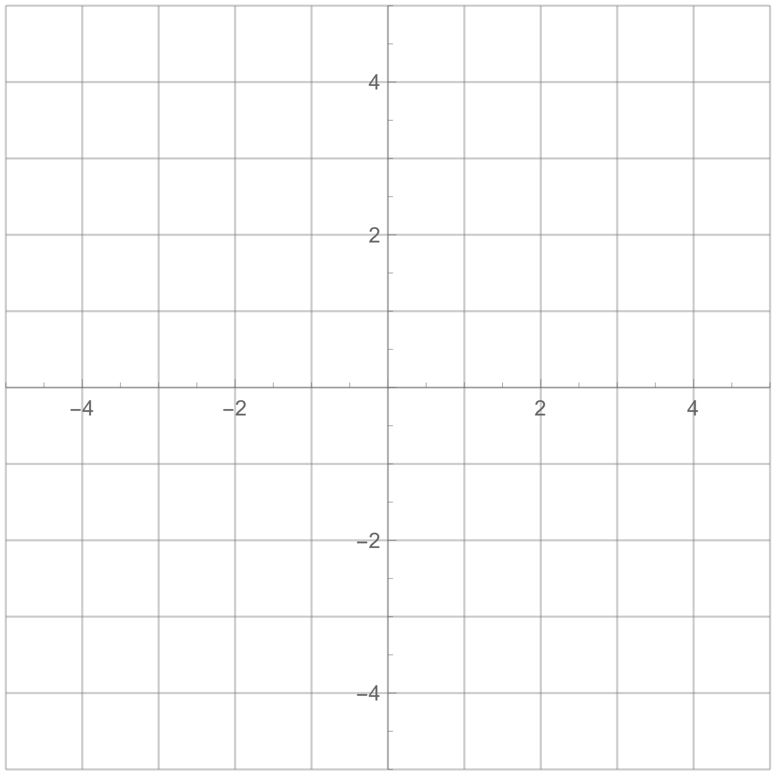
(b)

$$2x + 4y = 8$$



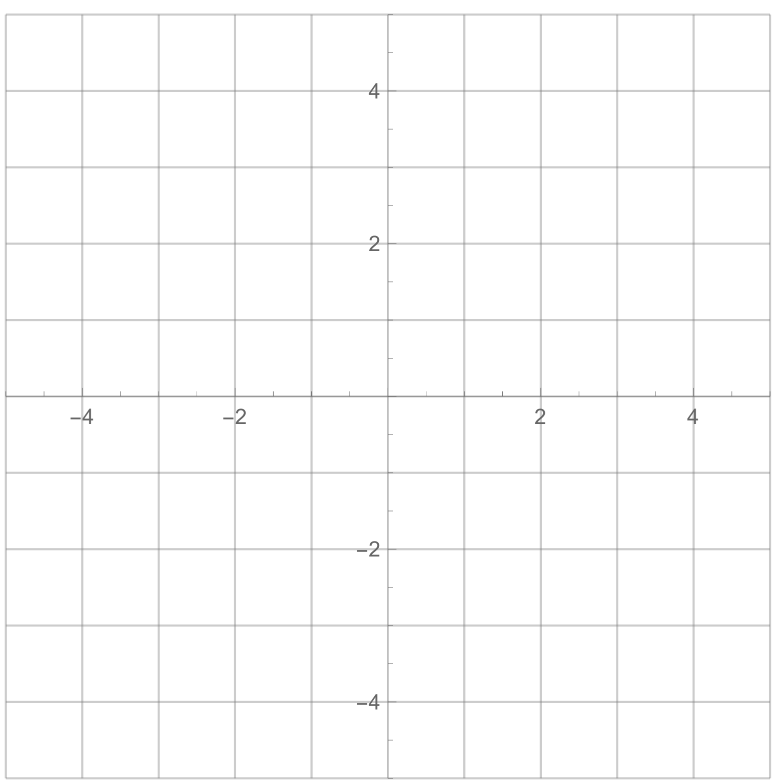
(c)

$$3x - 2y = 0$$



(d)

$$6x = 18$$



Section 4.1 - Systems of Linear Equations in Two Variables

Suppose we go to a movie theater and there are two packages for discounted tickets:
Package 1: 2 adult tickets and 1 child ticket for \$32
Package 2: 1 adult ticket and 3 child tickets for \$36

Based off of this information, can we figure how much the adult and child ticket discount prices are?

We can! To do this let A stand for the price of the adult ticket and let C stand for the price of the child ticket, then we get the following two equations from the two packages:

$$\begin{array}{rcrcr} 2A & + & C & = & 32 \\ A & + & 3C & = & 36 \end{array}$$

This is a system of two linear equations in two variables. To find the answer, we need to find a pair of numbers (A, C) which satisfy *both* equations simultaneously.

Definition 3 (System of Two Linear Equations in Two Variables)**.** *Given the* linear system

$$\begin{array}{rcrcr} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

where a, b, c, d, h, and k are real constants, a pair of numbers $x = x_0$ and $y = y_0$ (often written as an ordered pair (x_0, y_0)) is a solution *of this system if each equation is satisfied by the pair. The set of all such ordered pairs is called the* solution set *for the system. To* solve *a system is to find its solution set.*

There are a few ways we can go about solving this: *graphically*, using *substitution*, and *elimination by addition*.

Solving by Graphing. To solve this problem by graphing, we simply graph the two equations in the system, then find the intersection. Since we're relying on a graph to find this point, we need to check our solution in the equations of the system.

The blue line is the graph of $2A + C = 32$ and the purple line is the graph of $A + 3C = 36$. The red point is the intersection point $(12, 8)$. So the ticket prices are \$12 for an adult ticket and \$8 for a child ticket. Check the point $(12, 8)$ in both equations:

$$2A + C = 2(12) + 8 = 24 + 8 = 32 \quad \checkmark$$

and

$$A + 3C = 12 + 3(8) = 12 + 24 = 36 \quad \checkmark.$$

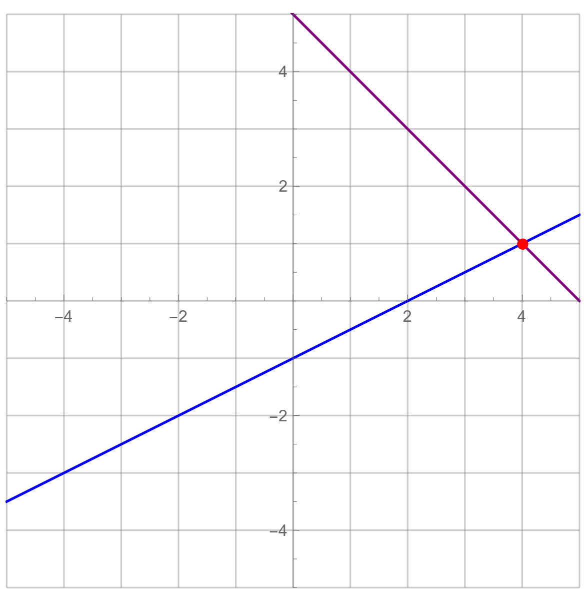
This verifies the solution.

There are actually 3 types of solutions to a system of linear equations

(1) Consider the system

$$\begin{array}{rcrcr} x & - & 2y & = & 2 \\ x & + & y & = & 5 \end{array}$$

If we graph the lines, we get

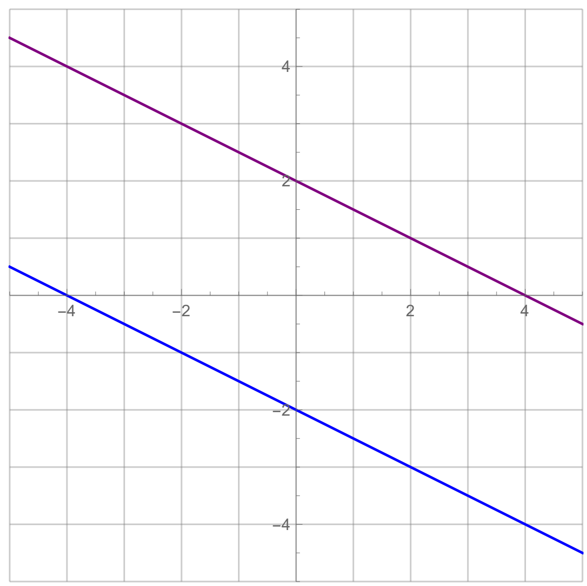


In this case, like before, we see only the *one solution* at $(4, 1)$. (You should check this in the system!)

(2) Consider the system

$$\begin{array}{rcrcl} x & + & 2y & = & 4 \\ 2x & + & 4y & = & 8 \end{array}$$

If we graph the lines, we get

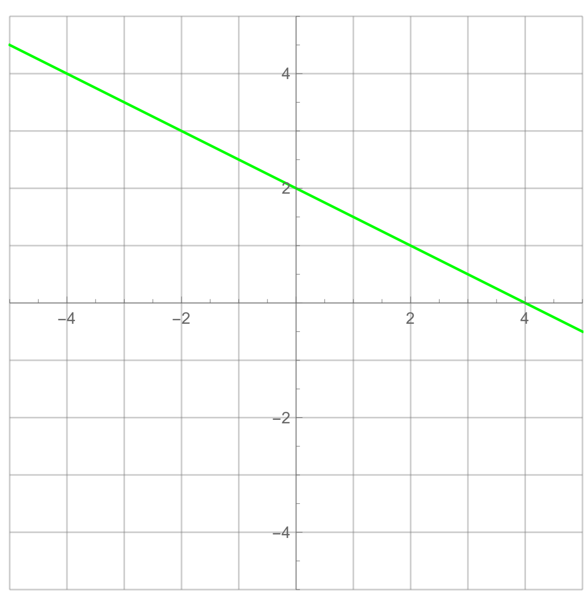


In this case, the lines are parallel and so they never intersect. In this case, there is *no solution*.

(3) Consider the system

$$\begin{array}{rcrcl} 2x & + & 4y & = & 8 \\ x & + & 2y & = & 4 \end{array}$$

If we graph the lines, we get



Here, both of the lines are exactly the same. In this case, there is an infinite number of solutions.

Definition 4. *A system of linear equations is called* consistent *if it has one or more solutions and* inconsistent *if it has no solutions. Further, a consistent system is called* independent *if it has exactly one solution (called the* unique solution*) and is called* dependent *if it has more than one solution. Two systems of equations are called* equivalent *if they have the same solution set.*

Theorem 1. *The linear system*

$$\begin{array}{rcrcl} ax & + & by & = & h \\ cx & + & dy & = & k \end{array}$$

must have

(1) *Exactly one solution (consistent and independent).*

(2) *No solution (inconsistent).*

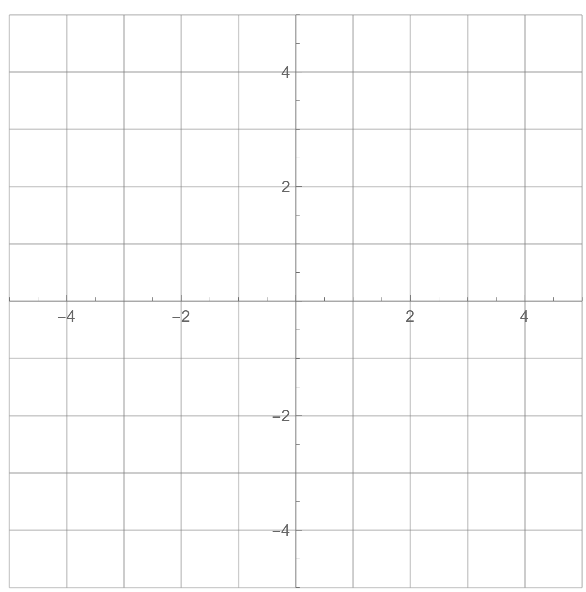
(3) *Infinitely many solutions (consistent and dependent).*

There are no other possibilities.

Example 4. *Solve the following systems of equations using the graphing method. Determine whether there is one solution, no solutions, or infinitely many solutions. If there is one solution, give the solution.*

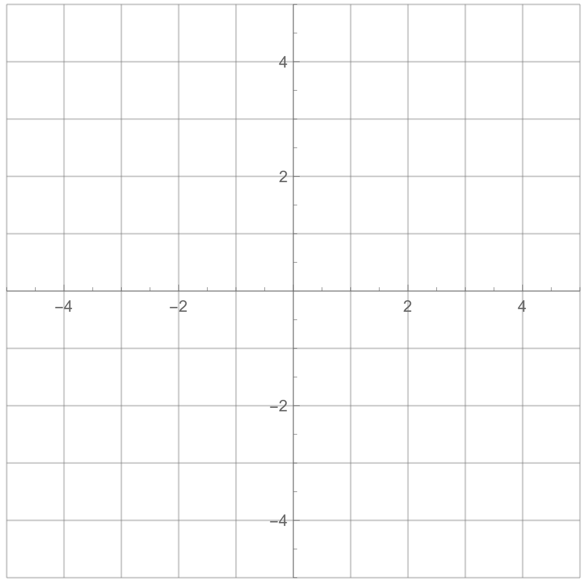
(a)

$$\begin{array}{rcrcl} x & + & y & = & 4 \\ 2x & - & y & = & 5 \end{array}$$



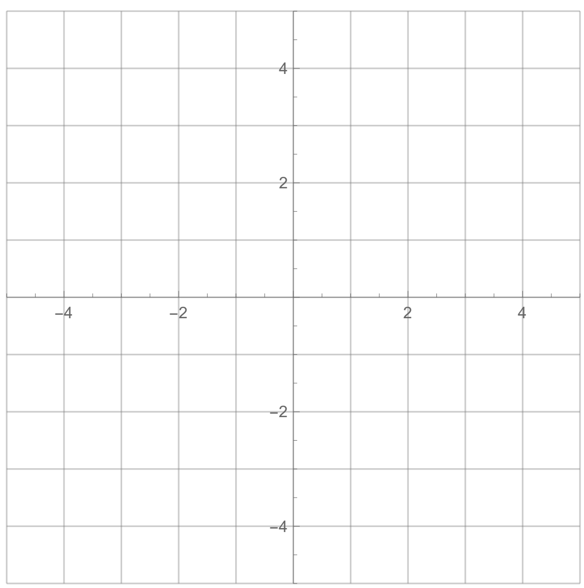
(b)

$$\begin{aligned} 6x - 3y &= 9 \\ 2x - y &= 3 \end{aligned}$$



(c)

$$\begin{aligned} 2x - y &= 4 \\ 6x - 3y &= -18 \end{aligned}$$



Solving by Substitution. When solving a system by substitution, we solve for one of the variables in one of the equations, then plug that variable into the other equation.

Example 5. *Solve the following system using substitution*

$$\begin{array}{rcrcl} 2x & - & y & = & 3 \\ x & + & 2y & = & 14 \end{array}$$

Solution. *Let's solve the first equation for y. To do this, we'll move the y to the right side, and the 3 to the left:*

$$2x - 3 = y$$

Then we plug this into the second equation for y:

$$x + 2(2x - 3) = 14$$

then we solve for x in this

$$x + 4x - 6 = 5x - 6 = 14$$

which gives

$$5x = 20$$

and so

$$x = 4.$$

Then we plug this into the equation we have for y to find that

$$y = 2(4) - 3 = 8 - 3 = 5$$

And so the solutions is $x = 4, y = 5$.

Example 6. *Solve the following system using substitution*

$$\begin{array}{rcrcl} 3x & + & 2y & = & -2 \\ 2x & - & y & = & -6 \end{array}$$

Solution. $x = -2, y = 2$

Solving Using Elimination. We now turn to a method that, unlike graphing and substitution, is generalizable to systems with more than two variables easily. There are a set of rules to follow when doing this

Theorem 2. *A system of linear equations is transformed into an equivalent system if*

(1) *two equations are interchanged*

(2) *an equation is multiplied by a nonzero constant*

(3) *a constant multiple of one equation is added to another equation.*

Example 7. *Solve the following system using elimination*

$$\begin{array}{rcrcr} 3x & - & 2y & = & 8 \\ 2x & + & 5y & = & -1 \end{array}$$

Solution. *If we subtract the second equation from the first one, we end up with the new system*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ 2x & + & 5y & = & -1 \end{array}$$

Now, we can subtract 2 *times the first equation* ($2x - 14y = 18$) *from the second equation to get*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & 19y & = & -19 \end{array}$$

Now we divide the second equation by 19 *to get*

$$\begin{array}{rcrcr} x & - & 7y & = & 9 \\ & & y & = & -1 \end{array}$$

and finally, we will add 7 *times the second equation* ($7y = -7$) *to the first equation*

$$\begin{array}{rrcr} x & & = & 2 \\ & y & = & -1 \end{array}$$

This gives the answer of $x = 2, y = -1$.

We could have also used a combination of substitution and elimination above, for example, once we knew that $y = -1$, we could have just plugged that into the first equation, but this solution was a little preview for the later sections.

Example 8. *Solve the system using elimination*

$$\begin{array}{rcrcr} 5x & - & 2y & = & 12 \\ 2x & + & 3y & = & 1 \end{array}$$

Solution. $x = 2, y = -1$